Journal of Basic and Applied Engineering Research
p-ISSN: 2350-0077; e-ISSN: 2350-0255; Volume 6, Issue 1; January-March, 2019, pp. 21-22

http://www.krishisanskriti.org/Publication.html

# Sampling and Reconstruction of Functions in Shift-invariant Spaces

**Anuj Kumar**

*Department of Mathematics Indian Institute of Technology Delhi*
*New Delhi 110016, India*
*E-mail: ak.maths.iitd@gmail.com*

**Abstract**

Modern digital signal processing always uses discrete samples which are obtained by sampling $f$ on a discrete set $X$. Thus it is natural to ask whether and how $f$ can be recovered from its samples. Let $V$ be a given class of signals on $\mathbb{R}$. We are interested to find the conditions on the sampling set $X = \{x_k : k \in \mathbb{Z}\}$ such that every $f \in V$ can be reconstructed uniquely and stably from its samples $\{f(x_k) : x_k \in X\}$, i.e., there are constants $0 < A \leq B < \infty$ such that

$$A\|f\|_{L^2(\mathbb{R})} \leq \Big(\sum_{k\in\mathbb{Z}} |f(x_k)|^2\Big)^{1/2} \leq B\|f\|_{L^2(\mathbb{R})}, \quad \forall f \in V.$$

In practice, the measurement apparatus gives only local averages of $f$ near certain points. Precisely, measured sample values are as follows

$$\langle f, u_k\rangle = \int_{\mathbb{R}} f(x)u_k(x)dx,$$

where $\{u_k : k \in \mathbb{Z}\}$ is a sequence of averaging functions satisfying the following conditions:

$$(i)\ \ \text{supp}\ u_k \subset \Big[x_k - \tfrac{\delta}{2}, x_k + \tfrac{\delta}{2}\Big], \qquad u_k \geq 0 \ \ and \ \ (ii) \ \ \int_{\mathbb{R}} u_k(x)dx = 1.$$

The aim of this talk is to discuss about average sampling and reconstruction in shift-invariant spaces [2] $V(\varphi)$ defined by

$$V(\varphi) := \Big\{f(x) = \sum_{k\in\mathbb{Z}} c_k\varphi(x-k) : c = (c_k) \in \ell^2(\mathbb{Z})\Big\}$$

where $\varphi \in L^2(\mathbb{R})$ satisfies

$$E\|c\|_{\ell^2(\mathbb{Z})} \leq \Big\|\sum_{k\in\mathbb{Z}} c_k\varphi(\cdot - k)\Big\|_{L^2(\mathbb{R})} \leq F\|c\|_{\ell^2(\mathbb{Z})} \qquad \forall\ c = (c_k) \in \ell^2(\mathbb{Z})$$

for some $E, F > 0$. These spaces play an important role in multi resolution analysis, sampling theory and several other research areas of signal and image processing [1, 2]. It is well-known [4] that if $\varphi$ has moderate decay in time domain, then $V(\varphi)$ is a reproducing kernel Hilbert space. Thus, the point-wise evaluation is continuous on $V(\varphi)$. We discuss the reconstruction of signals in $V(\varphi)$ from local average samples. Consider a class $\mathcal{F}$ of continuously differentiable functions $\varphi$ defined on $\mathbb{R}$ satisfying the following conditions:

(a) There exist constants $C_1, C_2 > 0$ and $\alpha > 0.5$ such that $|\varphi(x)| \leq \frac{C_1}{|x|^\alpha}$ and $|\varphi'(x)| \leq \frac{C_2}{|x|^\alpha}$ for sufficiently large $x$,

(b) $\operatorname*{ess\,sup}_{\omega\in[0,1]} \sum_{k\in\mathbb{Z}} (\omega+k)^2|\widehat{\varphi}(\omega+k)|^2 < \infty$,

(c) $\left\{\varphi(\cdot - k) : k \in \mathbb{Z}\right\}$ forms a Riesz basis for $V(\varphi)$.

Let $B = \operatorname*{ess\,sup}_{\omega\in[0,1]} B(\omega)$, where

$$B(w) := \frac{\sum_{k\in\mathbb{Z}} (\omega+k)^2|\widehat{\varphi}(\omega+k)|^2}{\sum_{k\in\mathbb{Z}} |\widehat{\varphi}(\omega+k)|^2}.$$

We prove that if $\varphi \in \mathcal{F}$ and a sampling set $X = \{x_k : k \in \mathbb{Z}\}$ such that $\cdots < x_k < x_{k+1} < \cdots$ and $\sup_{k\in\mathbb{Z}}(x_{k+1} - x_k) = \beta < \frac{1}{2\sqrt{B}}$, then every $f \in V(\varphi)$ can be reconstructed uniquely and stably from its local averages near $x_k$ provided the support length of averaging functions $\delta$ is less than $\frac{1}{2\sqrt{B}} - \beta$.

For the definition of frames and Riesz basis, we refer [3]. Define $\widetilde{\varphi}$ by

$$\widehat{\widetilde{\varphi}}(\omega) := \frac{\widehat{\varphi}(\omega)}{\sum_{k\in\mathbb{Z}} |\widehat{\varphi}(w+k)|^2},$$

then it is well-known that $\left\{\widetilde{\varphi}(\cdot - k) : k \in \mathbb{Z}\right\}$ is a dual Riesz basis for $V(\varphi)$. For each $k \in \mathbb{Z}$, define $g_k = \sum_{m\in\mathbb{Z}} \langle u_k, \widetilde{\varphi}(\cdot - m)\rangle \varphi(\cdot - m)$, then $\langle f, g_k\rangle = \langle f, u_k\rangle$ holds for any $f \in V(\varphi)$. Moreover, $\left\{\left(\frac{x_{k+1}-x_{k-1}}{2}\right)^{1/2} g_k : k \in \mathbb{Z}\right\}$ is a frame for $V(\varphi)$ with frame bounds $M = \frac{2}{81}\left(1 - 2(\beta+\delta)\sqrt{B}\right)^3$ and $N = \left(1 + \pi\sqrt{2B}(\delta+\beta)\right)^2$. Thus, every $f \in V(\varphi)$ can be reconstructed from its local averages by applying the following iterative frame reconstruction algorithm

$$\begin{aligned} Sf &:= \frac{2}{M+N} \sum_{k\in\mathbb{Z}} \left(\frac{x_{k+1}-x_{k-1}}{2}\right) \langle f, u_k\rangle\, g_k, \\ f_0 &= Sf, \\ f_{n+1} &= f_n + S(f - f_n), \qquad n \geq 0. \end{aligned}$$

The error estimate after $n^{th}$ iteration is given by

$$\|f - f_n\|_{L^2(\mathbb{R})} \leq \left(\frac{N-M}{N+M}\right)^{n+1} \|f\|_{L^2(\mathbb{R})}.$$

The detailed proof of the above results can be found in [5]. We also illustrate the theoretical results numerically by considering $\varphi$ as $B$-spline function and Meyer scaling function. This is a joint work with Dr. Sivananthan Sampath.

# References

[1] A. Aldroubi. Non-uniform weighted average sampling and reconstruction in shift-invariant and wavelet spaces. *Appl. Comput. Harmon. Anal.*, 13(2):151–161, 2002.

[2] A. Aldroubi and K. Gröchenig. Nonuniform sampling and reconstruction in shift-invariant spaces. *SIAM Review*, 43(4):585–620, 2001.

[3] O. Christensen. *An introduction to frames and Riesz bases*, volume 7. Boston(MA):Birkhäuser, 2003.

[4] S.H. Kulkarni, R. Radha, and S. Sivananthan. Non-uniform sampling problem. *J. Appl. Funct. Anal.*, 4(1):58–74, 2009.

[5] A. Kumar and S. Sampath. Average sampling and reconstruction in shift-invariant spaces and variable bandwidth spaces. *Appl. Anal., DOI: 10.1080/00036811.2018.1508652*, 2018.